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Effect of sinusoidal and non-sinusoidal excitations on Rossler system

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Abstract
We analyze numerically the effect of sinusoidal and non-sinusoidal periodic forces in the ubiquitous Rossler system. The external sinusoidal and non-sinusoidal periodic forces considered are sine wave, rectified sine wave, modulus of sine wave, square wave, symmetric saw-tooth wave and asymmetric saw-tooth wave. Through numerical simulation which include bifurcation diagram, phase portrait, Poincare map, time series and maximal Lyapunov exponent, we find the occurrence of period-doubling bifurcation, chaos, intermittency, periodic windows, reverse period doubling bifurcation and various shapes of strange attractors due to the applied sinusoidal and non-sinusoidal forces in this system. A comparative study of the effect of various forces on above said nonlinear phenomena is also performed.

Keywords: Rossler system, Sinusoidal forces, Non-sinusoidal forces, Bifurcation, Chaos.

1. Introduction
In the past few decades, much progress has been made in the interdisciplinary field of nonlinear chemical dynamics (NCD). Nonlinear chemical dynamics is the study of how systems whose behavior depends in a nonlinear fashion on the values of key variables like concentration in a chemical reaction, evolve in time [1-7]. Its applications can be found nearly in all fields of chemistry as well as engineering, mathematics, physics, biology and astronomy. Perturbation methods have proven to be useful in characterizing reaction mechanisms, where valuable information can be collected as the system relaxes from a perturbed state to its stable state in the concentration phase-space [8-12]. It is important to study the effect of sinusoidal and non-sinusoidal external periodic forces, because periodic forces can be easily generated and implemented in many experimental systems. By means of comparative study of the effect of different forces, we know the role of the shape of the forces on nonlinear chemical systems and to choose the suitable forces in creating and controlling various nonlinear behaviours. Recently, many studies have shown that the effect of different kinds of periodic forcing on nonlinear dynamical systems is considerable and they can change the dynamical behaviors drastically. For example, stochastic resonance and nonescape dynamics with different periodic forces [13,14], anti-control of chaos by certain periodic forces [15], generation of chaotic behavior by a distorted force [16], onset of homoclinic chaos by rectified and modulated sine forces [17], homoclinic bifurcation and chaos by an amplitude modulated force [18] and narrow band frequency modulated force [19] have been reported.
Motivated by the above considerations, in the present work, we wish to analyze numerically the effect of sinusoidal and non-sinusoidal periodic forces in Rossler system which are chemically and physically important systems. Compared with previous references, the dynamics of the forced system are analyzed with amplitude variation and the occurrence of period-doubling, strange attractor are not only demonstrated by Lyapunov exponent spectrum, bifurcation diagram but also verified by phase portrait, Poincare section and time series. It is organized as follows. The mode of sinusoidally and non-sinusoidally forced Rossler system is presented in section II. The mathematical form of various periodic forces considered in our present work is presented in section III. We numerically analyzed the bifurcation structures due to the applied sinusoidally and non-sinusoidally periodic forces such as sine wave, rectified and modulus of sine waves and symmetric, asymmetric saw-tooth waves and square-wave in section IV. The dynamical behaviours such as various routes to chaos, various bifurcations of chaos, periodic windows and reverse period-doubling bifurcations are analyzed

by calculating the Lyapunov exponents, bifurcation diagrams, time series, phase portraits and Poincare sections. Finally, we summarize the results and indicate the future directions in section V.

2. The Forced Rossler System

A simple model motivated by the dynamics of chemical reactions in a stirred tank is the following set of equations proposed by Rossler [20]. The general form of these equations serve as a simple model for many physical and chemical systems

$$\dot{x} = -(y+z) + F(t)$$
$$\dot{y} = x + ay \qquad (1)$$
$$\dot{z} = b + z(x-c)$$

where *a*, *b* and *c* are constant parameters. *F(t)* is an external time dependent sinusoidal and non-sinusoidal periodic forces. Complex dynamical behaviors, such as periodic, quasiperiodic and chaotic oscillations have been observed in both continuous flow stirred tank reactors [21, 22] and as transients in batch reactors [23]. Since then many researchers were attracted to this system, and made contributions to the study of the mechanism of complex phenomenon observed from experiments and numerical simulations. For instance, Yang *et al.* [24] showed a new proof for existence of horseshoe in the Rossler system. Localization of periodic orbits of the Rossler system under variation of its parameters is reported by Starkov *et al.* [25]. Ganesio *et al.*, [26]. Studied global qualitative view of bifurcation and dynamics in the Rossler system. Fei *et al.* [27] analyzed the stochastic period doubling bifurcation in a bounded Rossler system. Shinn [28] studied the chemical mechanism behind oscillations of the system and compared with Rossler system. In a very recent work, Barrio *et al.* [29] reported bifurcation of equilibria in Rossler model both qualitatively and numerically.

By analyzing the unperturbed Rossler system, we can observe two different equilibrium points:

$$X_+, X_- : (x^*, y^*, z^*) = \left(\frac{c \pm \sqrt{c^2-4ab}}{2}, \frac{-c \mp \sqrt{c^2-4ab}}{2a}, \frac{c \pm \sqrt{c^2-4ab}}{2a} \right)$$

If $x < c$, z(t) settles down near to the value $b / (c-x)$ (and rapidly, if $(c-x)$ is large), whereas if $x > c$, $z(t)$ increases exponentially. Thus the only nonlinearity in the dynamics involves this lifting and reinjection of the z(t) dynamics relative to the linear *(x,y)* motion [30, 31]. Period-1, 2 and 4 motions may be found to occur for c = 2.6, 3.5 and 4.1 respectively, which clearly shown in fig.1. Chaotic motion may be found to occur for $c > 4.3$. For a = 1/5, b = 1/5 and c = 5.7 the unperturbed Rossler system has been two equilibrium points X_+: (x^*,y^*,z^*) = (5.6929, ~28.4649, 28.4649), X_- = (x^*,y^*,z^*) = (0.0070, -0.0351, 0.0351) and all of these are unstable. The trajectories make a number of spirals around X_+, X_- which is shown in fig.2.

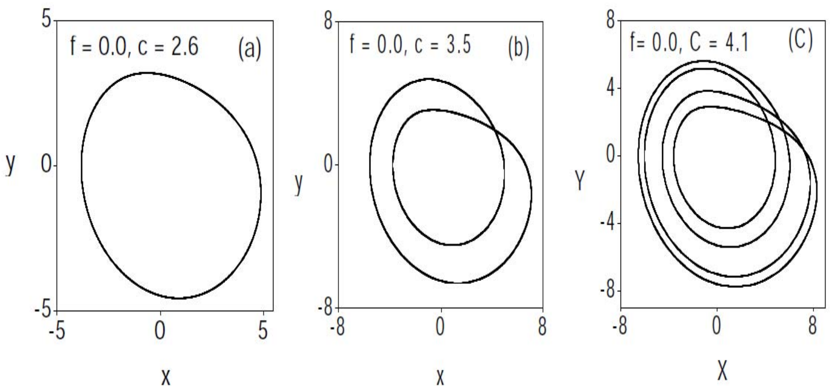

FIG. 1: Phase portraits of (a) period-T (for c = 2.6) (b) period-2T (for c = 3.5) and (c) period-4T (for c = 4.1) attractors of the Rössler equations (Eq.1) without perturbation ie.,$F(t) = 0$. The other parameter values are $a = 1/5$ and $b = 1/5$.

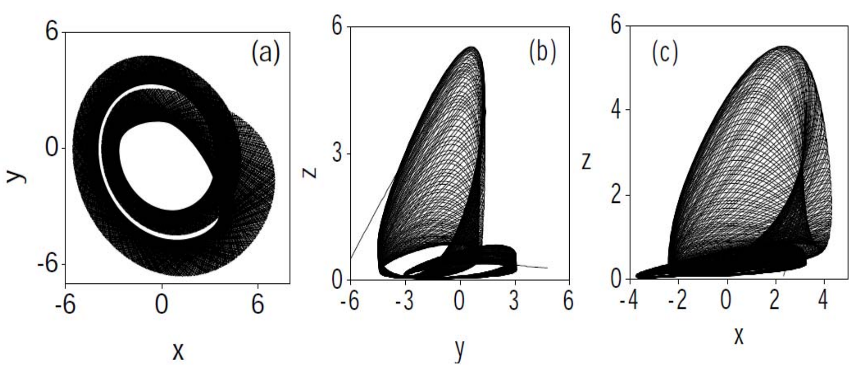

FIG. 2: Phase portraits of the chaotic attractor of the Rössler equations (Eq.1) without perturbation ie.,$F(t) = 0$ for the parameter values $c = 5.7$, $a = 1/5$ and $b = 1/5$.

Types of Sinusoidal and Non-sinusoidal Periodic Forces
The sinusoidal and non-sinusoidal periodic forces of our interest are: (i) Sine wave (ii) Rectified sine wave (iii) Modulus of sine wave (iv) Symmetric saw-tooth wave (v) Asymmetric saw-tooth wave (vi) Square wave. The shapes of the above said periodic forces are shown in fig.3. Mathematical form of sinusoidal and non-sinusoidal periodic forces such as sine wave, square wave, symmetric saw-tooth wave, asymmetric saw-tooth wave, rectified sine wave and modulus of sine wave are the following

$$F_{\sin}(t) = F_{\sin}(t + 2\pi/\omega) = f \sin \omega t. \quad (2)$$

$$F_{sq}(t) = F_{sq}(t + 2\pi/\omega) = \begin{cases} f, & 0 < t < \pi/\omega \\ -f, & \pi/\omega < t < 2\pi/\omega, \end{cases} \quad (3)$$

$$F_{sst}(t) = \begin{cases} \frac{4ft}{T_0}, & 0 < t < \pi/2\omega \\ -\frac{4ft}{T_0} + 2f, & \pi/2\omega < t < 3\pi/2\omega \\ \frac{4ft}{T_0} - 4f, & 3\pi/2\omega < t < 2\pi/\omega, \end{cases} \quad (4)$$

$$F_{ast}(t) = \begin{cases} \frac{2ft}{T_0}, & 0 < t < \pi/\omega \\ \frac{2ft}{T_0} - 2f, & \pi/\omega < t < 2\pi/\omega, \end{cases} \quad (5)$$

$$F_{msi}(t) = f|\sin(\omega t/2)|. \quad (6)$$

$$F_{rsi}(t) = \begin{cases} f, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega. \end{cases} \quad (7)$$

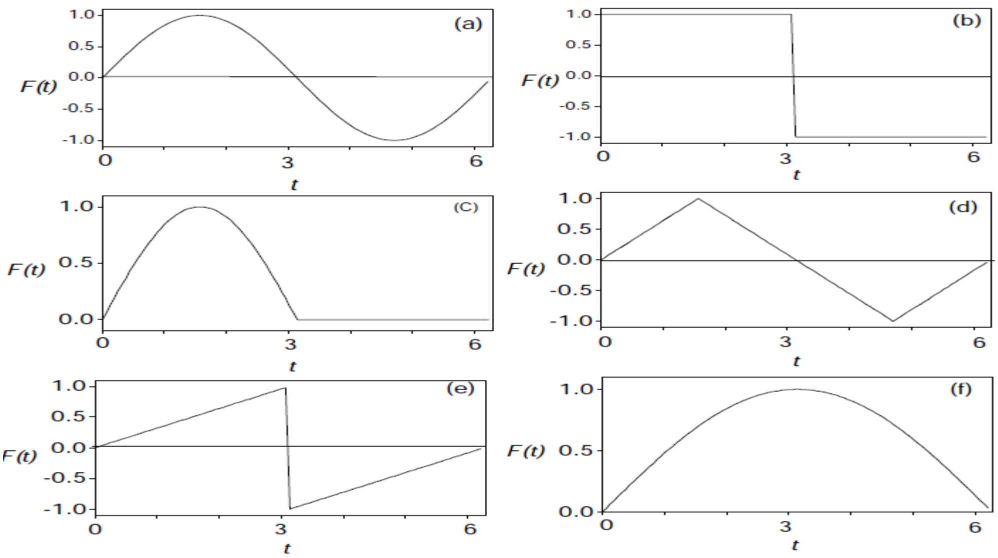

FIG. 3: Wave form of various periodic forces (a) sine wave (b) square wave (c) rectified sine wave (d) symmetric saw-tooth wave (e) asymmetric saw-tooth wave (f) modulus of sine wave. For all the forces period is $2\pi/\omega$ and amplitude $f = 1$.

3. Dynamics of the Sinusoidally and Non-sinusoidally Forced Rossler System
In this section, we analyze the dynamics of Rossler system with external perturbation, that is, excitations of sinusoidal and non-sinusoidal periodic forces. The period of all the forces is set to $2\pi/\ \omega$. The Rossler system is solved with different periodic forces by the fourth order Runge-Kutta method with time step size $(2\pi/\ \omega)/100$. Numerical solution corresponding to the first 500 drive cycles is left as transient. For each force the response of the system is studied by varying the amplitude *f* of it.

(i) Period-Doubling Cascades Due to Different Periodic Forces
First, we analyze the periodic behaviours of Rossler system (Eq.1) driven by different sinusoidal and non-sinusoidal periodic forces. We fix the parameters as $a = 0.2$, $b = 0.2$, $c = 5.7$ and $\omega = 1.0$. Figure 4 shows the bifurcation diagram and the corresponding maximal Lyapunov exponent (λ_m) for various periodic forces. We numerically calculated the maximal Lyapunov exponent (λ_m) employing Wolf *et al.* [32] algorithm. The Wolf *et al.* algorithm has been often used to compute Lyapunov exponents λ_m. For the periodic region (λ_m)

is found to be negative, at bifurcation (λ_m) = 0 and in the chaotic region (λ_m) is positive. When the forcing amplitude *f* is varied, we found many similarities and differences in the bifurcation pattern which is clearly seen in fig.4. In fig.4(a) with the force being $f \sin \omega t$ at f =2.3743, the period-*T* orbit becomes unstable and gives birth to period -2*T* orbit. Further increase in the value of *f* leads to successive bifurcations with period- 4*T*, 8*T* and 16*T* orbits are found to occur at f = 2.4632, 2.4863 and 2.4869 and which is accumulated at f = 2.48783, where the onset of chaos is observed. When the force is replaced by other sinusoidal and non-sinusoidal forces similar behavior is found to occur. In the case of square-wave, the period -2*T*, 4*T*, 8*T* and 16*T* orbit occurred at f = 1.72716, 1.73122, 1.73184 and 1.73197. At f = 1.732124, onset of chaos is observed. For other forces the critical values of *f* of different bifurcations are given in table 1. Magnification of a part of bifurcation diagram are shown in fig.5. From the table 1 and fig.5, period-doubling phenomenon is initiated to a lower value of *f* for square-wave while it is maximum for modulus of sine wave. For the force of asymmetric saw-tooth wave, the perturbed Rossler system enters a narrow co-existence of region of period-*T*, chaotic motion and period-doubling cascades as the value of *f* increases from f = 2.90. Figure 6 shows the phase portraits for the Rossler system (Eq.1) at the accumulation of period-doubling cascades (onset of chaos) for different sinusoidal and non-sinusoidal periodic forces. Examples of time evolution of the system (Eq.1) driven by sine wave force with different *f* values are shown in fig.7. The system (Eq.1) exhibits qualitatively different behaviour with periodic oscillations in fig.7(a), 7(b) and 7(c) and chaos in Fig.7(d).

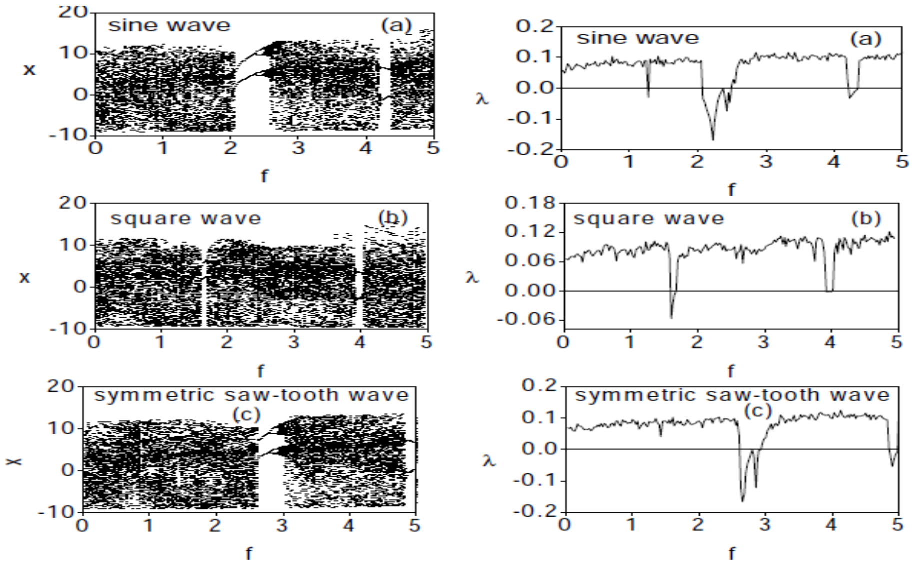

FIG. 4: Bifurcation diagrams and the corresponding maximal Lyapunov diagrams of Rössler system driven by various sinusoidal and non-sinusoidal periodic forces. The parameter values are fixed at $a = 0.2$, $b = 0.2$, $c = 5.7$ and $\omega = 1.0$.

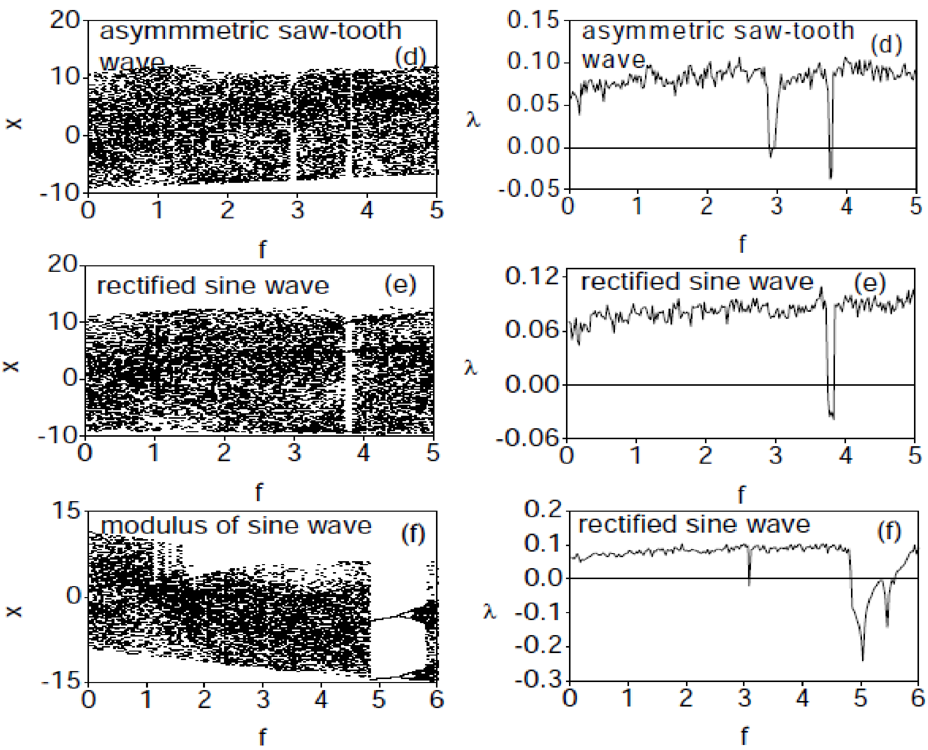

FIG. 4: Continued....

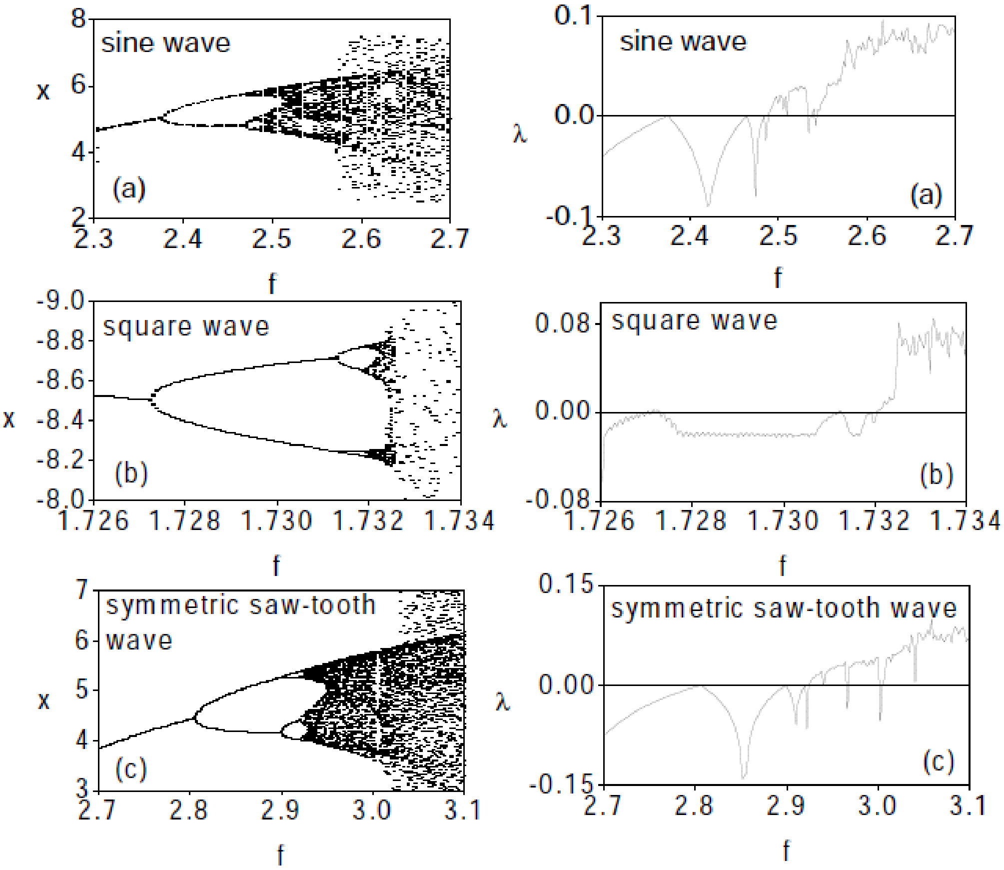

FIG. 5: Magnification of a part of bifurcation diagrams in fig.4

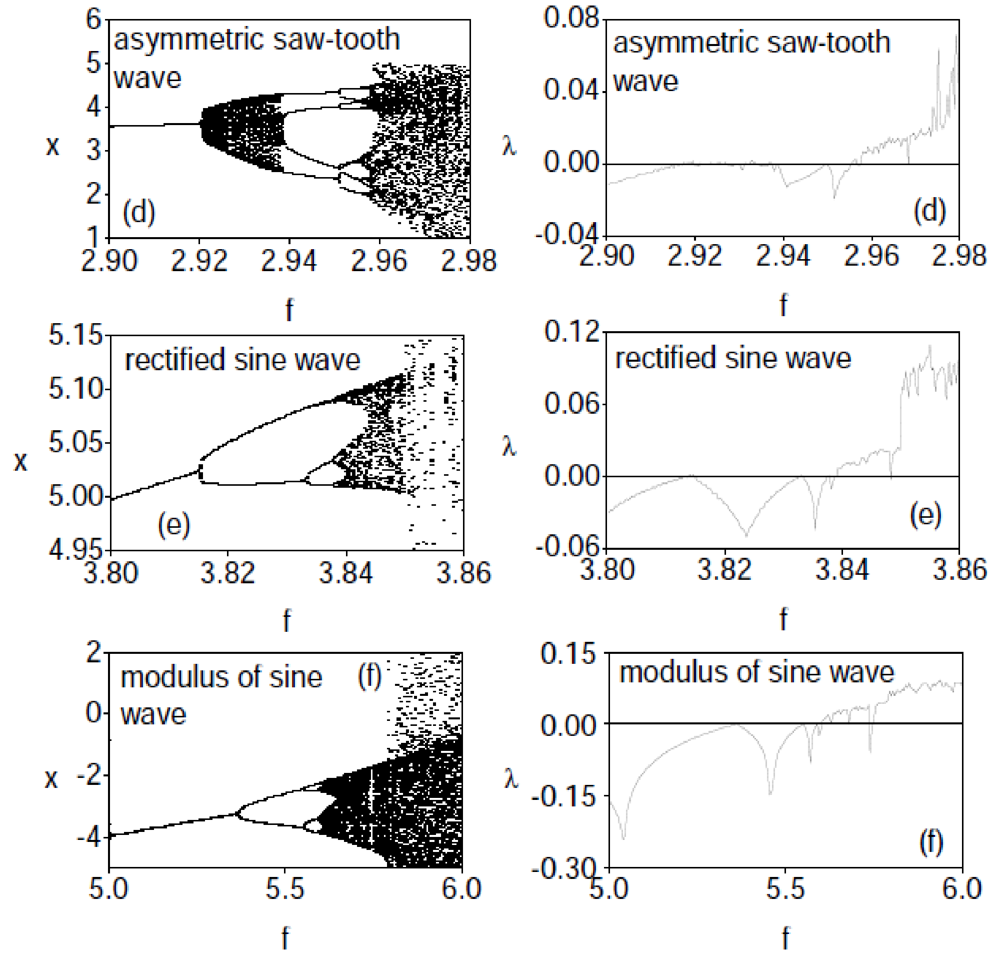

FIG. 5: Continued....

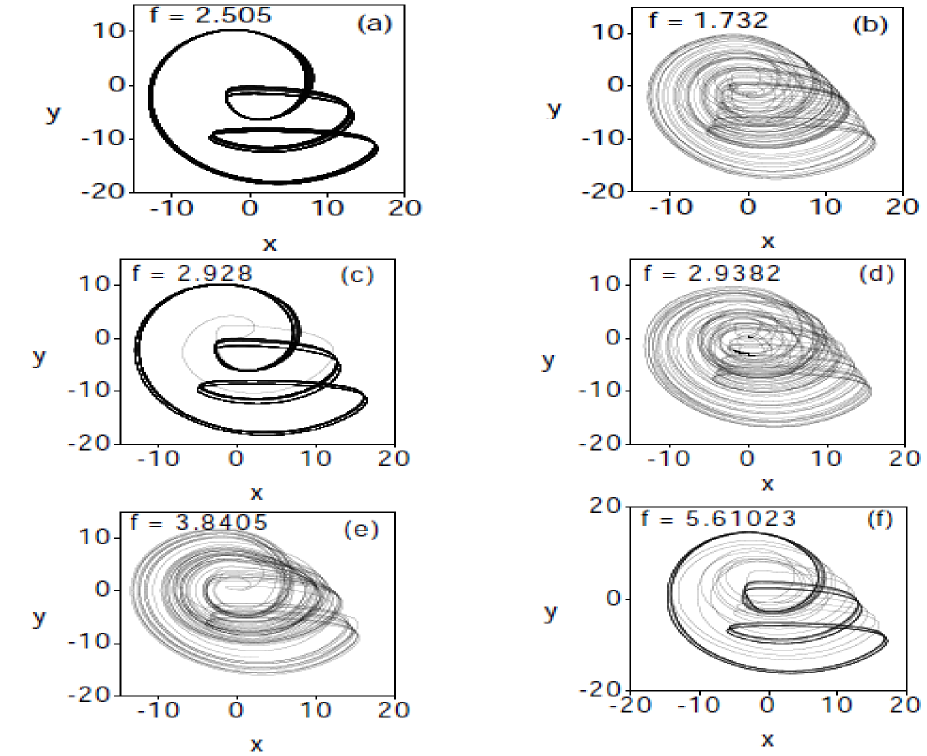

FIG. 6: Phase portraits of perturbed Rössler system (Eq.1) for different sinusoidal and non sinusoidal periodic forces at the accumulation of period-doubling bifurcations (onset of chaos (a) sine wave (b) square wave (c) symmetric saw-tooth wave (d) asymmetric saw-tooth wave (e rectified sine wave and (f) modulus of sine wave.

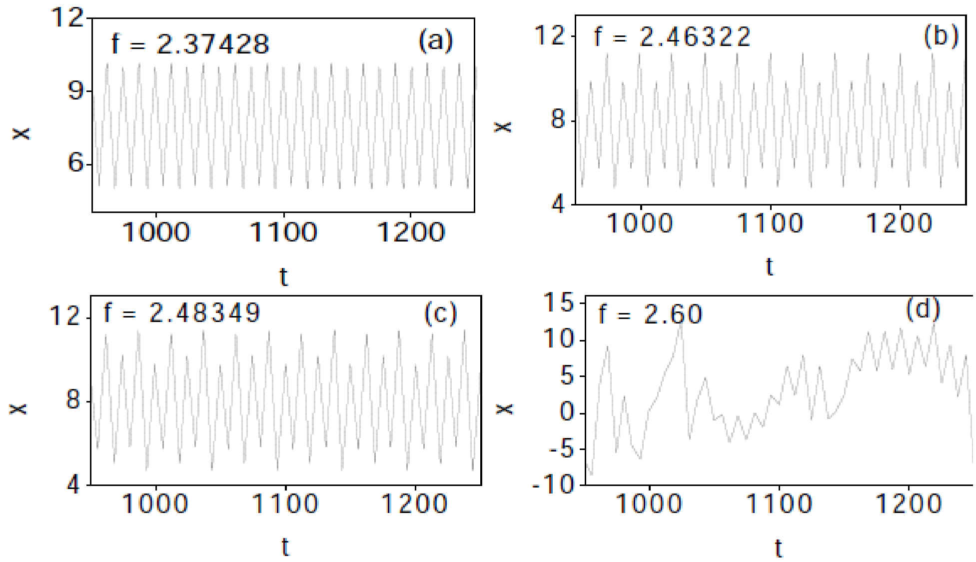

FIG. 7: Time evolution of the Rössler system (Eq.1) driven by sine wave force (a) Period-T (b) Period-2T (c) Period-4T and (d) Chaos.

(ii) Strange attractors due to the applied sinusoidal and non-sinusoidal periodic forces

In this section we numerically analyze the chaotic behavior of the system (Eq.1) driven by different forms of sinusoidal and non-sinusoidal periodic forces. When the forcing amplitude f is increased beyond the onset of chaos, the dynamics of the system is not fully chaotic alone but many fascinating changes in the dynamics takes place at different critical values of f. Particularly the chaotic orbits are interspersed by periodic windows, intermittent chaos, reverse period-doubling bifurcations, reverse period-3 bubble orbits and bifurcations of chaos which include band-merging, sudden widening, sudden destruction crises. This type of behavior is observed in this system (Eq.1) for all the sinusoidal and non-sinusoidal periodic forces.

(a) Crises and chaos: First we consider the system (Eq.1) driven by the force $f \sin \omega t$. Figure 8(a) shows the bifurcation diagram of the Rossler system (Eq.1) for $f \in [2.50, 2.52]$. For values of f slightly below $f = f_{bm} = 2.51333$ the chaotic attractor consists of two bands. The size of both the bands increase smoothly with the increase in the values of the control parameter f and the band merge together and form a single band chaotic attractor at $f = f_{bm}$. This is known as band merging bifurcation. Similar behavior is observed for all the forces. The values of f of different sinusoidal and non-sinusoidal periodic forces at which band merging bifurcation is observed for the Rossler system (Eq.1) is given in table 1. When the Rossler system (Eq.1) is driven by $f \sin \omega t$ another type of bifurcation which is seen in the fig.8(b) is the occurrence of sudden widening or sudden increase in the size of the chaotic attractor at f_{sw} = 2.58719. When the force is replaced by other forces similar behavior is found to occur in the Rossler system (Eq.1). The bifurcations such as band-merging and sudden widening of chaotic attractors are also termed as *crises* since they represent sudden discontinuous changes in the chaotic attractor as the control parameter f is varied. The values of f of different sinusoidal and non-sinusoidal periodic forces at which sudden widening bifurcation are observed and are given in table 1.

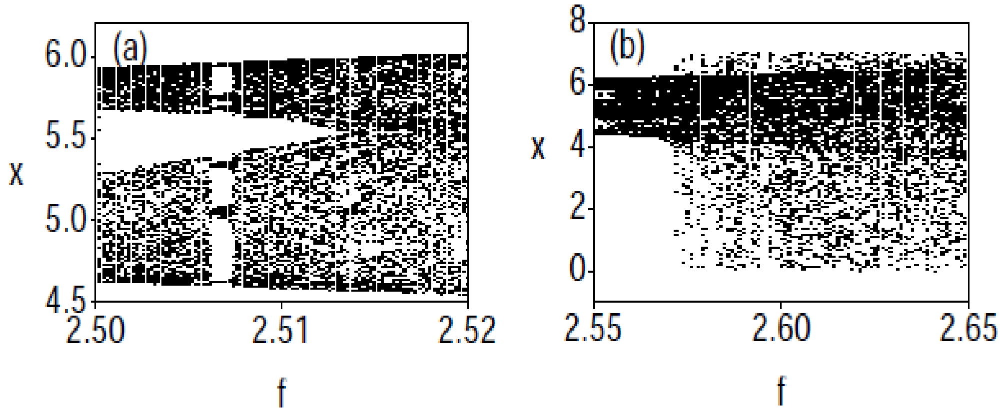

FIG. 8: Bifurcation diagrams of the Rössler system (Eq.1) driven by periodic sine wave force for showing the (a) band-merging (b) sudden widening bifurcations.

(b) Intermittency bifurcation: When a motion alternates randomly between long regular or laminar phases and short irregular bursts, it is said that the motion is intermittent or there is an intermittency

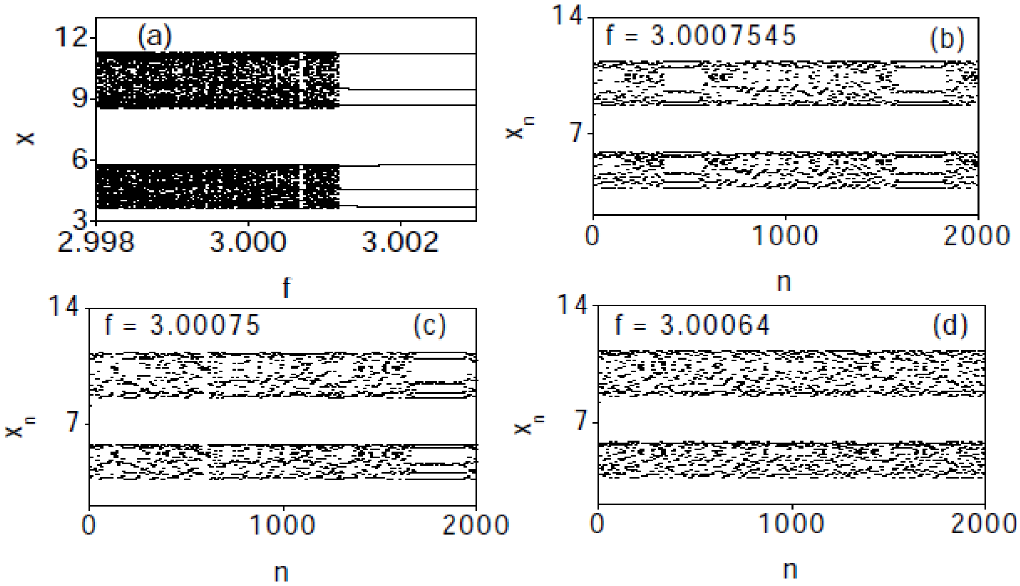

FIG. 9: ((a) Bifurcation diagram of the Rössler system (Eq.1) driven by symmetric saw-tooth wave in the intermittency region. (b-d) x(n) versus n, illustrating intermittency route to chaos.

In the intermittent route to chaos, initially the time series consists of the regular laminar motion interrupted by irregular bursts. As the control parameter f is increased, the length of the laminar region decreases and then bursts become frequent. First we show the occurrence of intermittency transition to chaos in the system (Eq.1) driven by a symmetric saw-tooth wave force. Fig. 9(a) shows the bifurcation phenomenon for $f \in [2.998, 3.003]$. We see that just above $f = f_I$ = 3.00015873, there is a stable period-10T while just below f_I there is chaos. Figure 9(b-d) shows x versus n where n is the time t in steps of $2\pi / \omega$ for three values of f. Intermittent dynamics is clearly seen in figs. 9b and 9c. We observed this type of behavior for other sinusoidal and non-sinusoidal periodic forces also but the period of the laminar region is different for different forces. The critical values of f of this behavior is presented in table-1.

TABLE I: Summary of bifurcation phenomena of the Rössler system (Eq. (1)) driven by various periodic forces with $a = 1/5$, $b = 1/5$, $c = 5.7$ and $\omega = 1$.

	Critical values of amplitude of the various forces					
Bifurcations	Sine wave	Square wave	Rectified sine wave	Symmetric saw-tooth wave	Asymmetric saw-tooth wave	Modulus of sine wave
Period-2T	2.37428	1.72716	3.81389	2.80413	2.92009	5.36914
Period-4T	2.46322	1.73122	3.83295	2.89725	*nil*	5.54312
Period-8T	2.48349	1.73184	3.83569	2.91799	2.95047	5.58655
Period-16T	2.48688	1.73196	3.83624	2.91836	2.95664	5.59026
Onset of chaos	2.48783	1.73312	3.83930	2.92582	2.95579	5.61119
Band-merging	2.51333	1.73238	3.84378	2.95052	2.95906	5.64399
Sudden widening	2.58719	1.73301	3.85098	3.03836	2.96148	5.83069
Intermittency	2.53334	2.65079	3.84781	3.00159	3.76719	5.74376

4. Conclusion

In this paper we numerically investigated the bifurcation of periodic orbits and strange attractors in Rossler system with different forms of sinusoidal and non-sinusoidal periodic forces. In our present work we fixed the period of forces as $2\pi/\omega$ and $\omega = 1$. In this system we have noticed several similarities and differences in the bifurcation structures in the presence of different sinusoidal and non-sinusoidal periodic forces.

It is found that the sinusoidal and non-sinusoidal forcing suppresses the original chaotic behaviour in some parameter ranges. The basic properties of the dynamics are analyzed by Lyapunov exponent diagram, bifurcation diagrams, time series, phase portraits and Poincare maps. With the change of amplitude variation f the sinusoidal and non-sinusoidal forced system exhibit limit cycles, pitchfork bifurcation, tangent bifurcation, period-doubling bifurcation, intermittency, crises and chaos. The system has more complex dynamical behaviours than that of the original system. Analytical methods such as multiple-scale perturbation and Melnikov techniques can be employed to the Rossler system to investigate certain nonlinear phenomena such as vibrational resonance, stochastic resonance, ghost resonance, horseshoe chaos, hysteresis etc. These will be studied in future.

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